Cash Flow Risk, Discounting Risk, and the Equity Premium Puzzle

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October 26, 2005

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Abstract

This article investigates the impact of cash flow risk and discounting risk on the aggregate equity premium. Our approach is based on the idea that consumption is hard to measure empirically, so if we substitute out an empirically difficult-to-estimate marginal utility by a pricing kernel of observables, we can evaluate the empirical performance of an equilibrium asset pricing model in a different way. Once the pricing-kernel process is specified, we can endogenously solve for the equity premium, the price of the market-portfolio and the term structure of interest rates within the same underlying equilibrium. Embedded in the closed-form solution are compensations for cash flow risk and discounting risk. With the solution for the risk premium explicitly given, we then calibrate the model to evaluate its empirical performance. This approach allows us to avoid the impact of the unobservable consumption or market portfolio on inferences regarding the model's performance. Our illustrative model is based on the assumption that aggregate dividend equals a fixed fraction of aggregate earnings plus noise, and the expected aggregate earnings growth follows a mean-reverting stochastic process. Moreover, the economy-wide pricing kernel is chosen to be consistent with (i) a constant market price of aggregate risk and (ii) a mean-reverting interest rate process with constant volatility. Estimation results show that the framework can mimic the observed market equity premium.

1 Introduction

In their seminal contribution, Mehra and Prescott (1985) show that the observed equity premium on the S&P 500 market index is far too high given the stochastic properties of aggregate consumption and under plausible assumptions about risk aversion. Furthermore, equity returns empirically covary little with aggregate consumption growth, implying also that the average equity premium can only be reconciled through an implausibly large coefficient of relative risk aversion. Table 1 in Mehra and Prescott (2003) documents that the average equity premium in the U.S. is 6.92%, while the real rate of interest is 1.14%, over the sample period of 1889-2000. Why have stocks delivered an average return of about 7% over risk-free bonds? Why is the observed real rate on Treasuries so low? Why is the systematic risk, as exemplified by the correlation between consumption growth and market-index return, so small?

Collectively known as the equity premium puzzle, this set of questions has consumed financial economists over the past two decades and generated competing explanations ranging from (i) generalizations to state-dependent utility functions (Constantanides (1990), Epstein and Zin (1991), Benartzi and Thaler (1995), Bakshi and Chen (1996), Campbell and Cochrane (1999), and Barberis, Huang, and Santos (2001)); (ii) the fear of catastrophic consumption drops (Reitz (1988)); (iii) the presence of uninsurable and idiosyncratic income risk (Heaton and Lucas (1996) and Mankiw (1986)); (iv) borrowing constraints (Constantinides, Donaldson, and Mehra (2002)); and (v) measurement errors and poor consumption growth proxies (Breeden, Gibbons, and Litzenberger (1989), Mankiw and Zeldes (1991), Ferson and Harvey (1992), and Aït-Sahalia, Parker, and Yogo (2004)). Despite the substantial research efforts, there is controversy whether these explanations can completely explain all aspects of the equity premium puzzle (Mehra and Prescott (2003)), and the original puzzle remains unsolved. That is, under plausible parameterizations, existing models can only generate a small equity premium.

This article expounds on a risk-based explanation without taking a stand on the precise parametric specification of the marginal utility function. Our approach is based on the idea that consumption is hard to measure empirically, so if we substitute out an empirically difficult-to-estimate marginal utility by a pricing-kernel function of observables we can evaluate the empirical performance of an equilibrium asset pricing model in a different way. That is, once the pricing-kernel process is specified, we can endogenously solve for the equity premium, the current price of the market portfolio and the term structure of interest rates within the same underlying equilibrium. Embedded in the closed-form solutions are compensations for cash flow risk and discounting risk. With these solutions for the risk premium, we can then calibrate the model to evaluate its empirical performance. This approach allows us to avoid the impact of unobservable consumption on inferences regarding an asset pricing model's performance.

We illustrate the potential of this modeling approach by using some simple assumptions. First, we posit that a fixed proportion of the market-portfolio earnings (plus some noise) will be paid out as dividends. This assumption allows us to directly link the stock price and the equity premium to the firm's earnings, instead of dividends. This modeling feature is important because dividend-based stock valuation models have not succeeded empirically, and investors are far more interested in the earnings of a stock rather than its dividends. Second, we assume some marginal utility function that is consistent with both a constant market price of aggregate risk and a single-factor Vasicek (1977) term structure of interest rates. It is further assumed that the market-portfolio earnings-per-share (EPS) obeys a proportional stochastic process, with its expected growth rate following a mean-reverting process (under the physical probability measure). Thus, in our equity valuation setting, there is an embedded stochastic term structure of interest rates, the expected EPS growth follows a stochastic process, the current market-index level depends on earnings (instead of dividends), and both cash flow risk and interest rate risk are priced. The rationale for our assumptions will be discussed in more details shortly.

It is shown that risk aversion implicit in the pricing kernel introduces a wedge between the physical process and the risk-neutralized process of variables in the economy. Specifically, the working of risk aversion makes the risk-neutral drift of the interest rate process higher than its physical counterpart and leads to a heavier discounting of stochastic cash flow streams. This mechanism generates lower market valuations and a higher equity premium (even though this effect also raises bond yields).

Risk aversion also affects the risk-neutralized cash flow process: the risk-neutral drifts for both the earnings and the expected earnings growth processes are lower than their counterpart under the physical probability measure. Such a mapping is suggestive of a positive compensation for both earnings risk and expected earnings-growth risk. Overall, the equity premium is a weighted sum of compensations for risks associated with interest rate, earnings, and expected earnings-growth shocks, with the weights dependent on the state-of-the economy and the structural parameters.

Our empirical implementation provides several insights on how discounting risk and cash flow risks are reflected and simultaneously priced in the S&P 500 index and default-free bonds. We find that the interest-rate risk premium is negative and it contributes to a 77.16 basis-point spread between the market-portfolio and the risk-free interest rate. Moreover, the compensation for expected earnings-growth risk is negligible, and the compensation for earnings risk is 6.53%. It is the risk premium for earnings uncertainty, and not expected earnings-growth uncertainty, that largely drives the equity premium. The total model-derived equity premium is 7.31% and quantitatively robust under perturbations to test design methods. Overall, our empirical exercise demonstrates that the signs of the risk premiums are consistent with economic theory and show promise in explaining the behavior of the average equity premium and the Treasury yield curve. We argue that replacing the marginal utility by a pricing-kernel function of observables, and sensibly parameterizing the discounting structure and cash flows, is crucial to achieving a reasonable equity premium and improved performance.

The purpose of this article is not to test whether a particularly parameterized economic model would be able to explain the observed equity premium under some reasonable set of parameter values. Rather, the goal is to show that given the unobservability of key economic variables (such as consumption and the market portfolio), an alternative approach to testing an economic model is to rely on its internal equilibrium relations to substitute out unobservable variables by functions of observable financial market variables. Then, a test on the resulting equilibrium relations amounts to a test on the economic model itself. Perhaps, another way to look at the results in this article is that it shows what basic properties an empirically successful pricing kernel must have in order to be consistent with the observed equity premium in the U.S. stock market.

In what follows, Section 2 outlines assumptions and develops analytical expressions for the price of the market portfolio and the equity premium. Section 3 describes the data on S&P 500 earnings, equity premium, interest rates, and the panel of bond prices. Section 4 estimates the valuation model and discusses its implication for the equity premium. Concluding statements are provided in Section 5. The mathematical derivations for the price of the market portfolio and the equity premium are provided in the Appendix.

2 Economic Determinants of Equity Premium

This section develops a framework to study the determinants of the time-t price of the market-portfolio, P_t, for each time $t \geq 0$, and the instantaneous market-index risk premium $\mu_t - r_t$, for short interest rate r_t.

Consider a continuous-time, infinite-horizon economy whose underlying valuation standard is represented by some pricing-kernel process, denoted by M_t. Assume that the market-portfolio entitles its holder to an infinite dividend stream $\{D_t : t \geq 0\}$. Asset pricing models under the perfect-markets assumption implies

$$
\begin{aligned}
P_t &= \int_t^\infty E_t\left[\frac{M_u}{M_t} D_u\right] du, \qquad \text{and,} && (1)\\
\mu_t - r_t &= -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dP_t}{P_t}\right)/dt, && (2)
\end{aligned}
$$

where $E_t[\cdot]$ is the time-t conditional expectation operator with respect to the objective probability measure. All variables in (1)-(2) are in nominal terms. In this framework, the instantaneous equity premium and the price of the market-portfolio are determined endogenously and jointly within the same underlying risk-return equilibrium. The basic model outlined below is adopted from Bakshi and Chen (2005).

2.1 Cash Flow Process

To explicitly solve (1)-(2), assume that the market-portfolio has a constant dividend-payout ratio (plus noise), α (with $1 \geq \alpha \geq 0$), that is,

$$D_t\, dt = \alpha\, Y_t\, dt \; + \; d\, Z_t, \tag{3}$$

where Y_t is the aggregate earnings-per-share (EPS) flow at t and hence $Y_t\, dt$ is the total EPS over the interval from t to $t + dt$, and $d\, Z_t$ is the increment to a martingale process with zero mean. The existence of $d\, Z_t$ allows the market-portfolio dividends to randomly deviate from the fixed proportion of its EPS, and it makes D_t and Y_t not perfectly substitutable. Although this temporary deviation could be correlated with recent earnings and past deviations, incorporating this feature, or the stochastic pay-out ratio feature, into the assumption would unnecessarily complicate the model (see Lintner (1956), Marsh and

Merton (1987), Barsky and Delong (1993), and Menzly, Santos, and Venonesi (2004)).

Under the objective probability measure, Y_t is assumed to follow a process given below:

$$\begin{aligned} \frac{dY_t}{Y_t} &= G_t\,dt + \sigma_y\,dW_t^y, && (4) \\ d\,G_t &= \kappa_g\left(\mu_g^* - G_t\right)\,dt + \sigma_g\,dW_t^g, && (5) \end{aligned}$$

for constants σ_y, κ_g, μ_g^* and σ_g. The long-run mean for both G_t and actual EPS growth $\frac{dY_t}{Y_t}$ is μ_g^*, and the speed at which G_t adjusts to μ_g^* is reflected by κ_g. Further, $\frac{1}{\kappa_g}$ measures the duration of the firm's business growth cycle. Volatility for both earnings growth and changes in G_t is time-invariant.

The cash flow process parameterized in (4) offers enough flexibility to model the level of the market-portfolio and the instantaneous equity premium (see also Bakshi and Chen (1997) and Longstaff and Piazzesi (2004)). First, both actual and expected earnings growth can take either positive or negative values, reflecting business cycles. Second, expected EPS growth G_t is mean-reverting and has both a permanent component (reflected by μ_g^*) and a transitory component, so that G_t can be high or low relative to its long-run mean μ_g^*. Finally, since Y_t is observable and G_t can be obtained from analyst estimates, we can learn about the equity premium based on readily identifiable and observable state variables.

2.2 The Discounting Process

Turning to the pricing kernel, assume, as in Constantinides (1992), that M_t follows an Ito process satisfying

$$\frac{dM_t}{M_t} = -r_t\,dt - \sigma_m\,dW_t^m, \tag{6}$$

for a constant σ_m, where the instantaneous discounting rate, r_t, follows the Ornstein-Uhlenbeck mean-reverting process:

$$d\,r_t = \kappa_r\left(\mu_r^* - r_t\right)\,dt + \sigma_r\,dW_t^r, \tag{7}$$

for constants κ_r, μ_r^* and σ_r. The pricing kernel can be interpreted in the context of the consumption-based asset pricing model. Suppose $M_t = C_t^{-\gamma}$ for coefficient of relative risk aversion γ and aggregate consumption C_t, then Ito's lemma impies $\frac{dM_t}{M_t} = -\gamma\frac{dC_t}{C_t} + \frac{1}{2}\gamma(1+\gamma)\left(\frac{dC_t}{C_t}\right)^2$. Thus, we can write risk-return equation (2) as $\mu_t - r_t = \gamma\,\mathrm{Cov}_t\left(\frac{dC_t}{C_t}, \frac{dP_t}{P_t}\right)/dt$, and the equilibrium $r_t\,dt = \gamma\,E_t\left(\frac{dC_t}{C_t}\right) - \frac{1}{2}(\gamma)(1+\gamma)\,E_t\left(\frac{dC_t}{C_t}\right)^2$. Thus, unlike the traditional approaches in Mehra and Prescott (1985) and Weil (1989), we independently model the interest rate dynamics as specified in (7).

Parameter κ_r measures the speed at which r_t adjusts to its long-run mean μ_r^*. The pricing kernel (6) leads to a single-factor Vasicek (1977) term structure of interest rates, that is, the τ-period bond-price is: $B(t,\tau) = \exp\left(-\xi[\tau] - \varsigma[\tau]r_t\right)$, where $\varsigma[\tau] \equiv \frac{1-e^{-k_r\,\tau}}{k_r}$, and $\xi(\tau) \equiv -\frac{1}{2}\,\sigma_r^2 \int_0^\tau \varsigma^2[u]\,du + \left(\kappa_r\,\mu_r\, + \mathrm{Cov}_t\left(\frac{dM_t}{M_t}, dr_t\right)\right)\int_0^\tau \varsigma[u]\,du$. This approach provides interest rate parameters that can be separately calibrated to the observed Treasury yield curve.

Notice that shocks to expected growth, W^g, may be correlated with both systematic shocks W^m and interest rate shocks W^r, with their respective correlation coefficients denoted by $\rho_{g,m}$ and $\rho_{g,r}$. In addition, the correlations of W^y with W^g, W^m and W^r are respectively denoted by $\rho_{g,y}$, $\rho_{m,y}$ and $\rho_{r,y}$. Thus, both actual and expected EPS growth shocks are priced risk factors. The noise process dZ_t in (3) is however assumed to be uncorrelated with G_t, M_t, r_t and Y_t, and hence it is not a priced risk factor.

2.3 Dynamics of the Market-Portfolio

Substituting assumptions (3)-(7) into (1)-(2), we can see that the conditional expectations in P_t must be a function of G_t, r_t and Y_t. Applying Ito's lemma to P_t and substituting the resulting expression into risk-return equation (2), we have the partial differential equation (PDE) for P_t (the details are given in the Appendix):

$$
\begin{aligned}
&\frac{1}{2}\,\sigma_y^2\,Y^2\,\frac{\partial^2 P}{\partial Y^2} + (G - \Pi_y)\,Y\,\frac{\partial P}{\partial Y} + \rho_{g,y}\sigma_y\sigma_g\,Y\,\frac{\partial^2 P}{\partial Y \partial G} + \rho_{r,y}\sigma_y\sigma_r\,Y\,\frac{\partial^2 P}{\partial Y \partial r} + \\
&\rho_{g,r}\sigma_g\sigma_r\,\frac{\partial^2 P}{\partial G \partial r} + \frac{1}{2}\sigma_r^2\,\frac{\partial^2 P}{\partial r^2} + \kappa_r\,(\mu_r - r)\,\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_g^2\frac{\partial^2 P}{\partial G^2} \\
&+\kappa_g\,(\mu_g - G)\,\frac{\partial P}{\partial G} - r\,P + \alpha\,Y = 0,
\end{aligned}
\tag{8}
$$

subject to the transversality condition $P_t < \infty$. The transversality condition states that the stock price stay bounded for all combinations of the parameters governing cash flows, discounting, and their risk premiums. In the valuation equation PDE (8) we set,

$$
\mu_g \equiv \mu_g^* - \frac{\Pi_g}{\kappa_g}, \tag{9}
$$

$$\mu_r \equiv \mu_r^* - \frac{\Pi_r}{\kappa_r}, \tag{10}$$

which are, respectively, the long-run means of G_t and r_t under the risk-neutral probability measure defined by the pricing kernel M_t. It can be shown that

$$\Pi_y \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dY_t}{Y_t}\right)/dt, \tag{11}$$

$$\Pi_g \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, dG_t\right)/dt, \tag{12}$$

$$\Pi_r \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, dr_t\right)/dt, \tag{13}$$

are the risk premium for the earnings shocks, expected earnings growth, and interest rate, respectively. Conjecture that the solution to the PDE (8) is of the form:

$$P_t = \alpha\, Y_t \int_0^\infty \overline{p}[t,u;G,r]\, du, \tag{14}$$

where $\overline{p}[t,u;G,r]$ can be interpreted as the time-t price of a claim that pays \$1 at a future date $t+u$. Solving the resulting valuation equation and the associated Ricatti equations subject to the boundary condition that $\overline{p}[t+u,0]=1$ yields,

$$\overline{p}[t,u;G,r] = \exp\left(\,\varphi[u] - \varrho[u]\, r_t + \vartheta[u]\, G_t\,\right), \tag{15}$$

where

$$\begin{aligned}
\varphi[u] \;\equiv\;& -\Pi_y\, u + \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(u + \frac{1-e^{-2\kappa_r u}}{2\kappa_r} - \frac{2(1-e^{-\kappa_r u})}{\kappa_r}\right) - \frac{\kappa_r\mu_r + \sigma_y\sigma_r\rho_{r,y}}{\kappa_r}\left(u - \frac{1-e^{-\kappa_r u}}{\kappa_r}\right)\\
&+\frac{1}{2}\frac{\sigma_g^2}{\kappa_g^2}\left(u + \frac{1-e^{-2\kappa_g u}}{2\kappa_g} - \frac{2}{\kappa_g}(1-e^{-\kappa_g u})\right) + \frac{\kappa_g\mu_g + \sigma_y\sigma_g\rho_{g,y}}{\kappa_g}\left(u - \frac{1-e^{-\kappa_g u}}{\kappa_g}\right)\\
&-\frac{\sigma_r\sigma_g\rho_{g,r}}{\kappa_r\kappa_g}\left(u - \frac{1}{\kappa_r}(1-e^{-\kappa_r u}) - \frac{1}{\kappa_g}(1-e^{-\kappa_g u}) + \frac{1-e^{-(\kappa_r+\kappa_g)u}}{\kappa_r+\kappa_g}\right),
\end{aligned} \tag{16}$$

$$\varrho[u] \equiv \frac{1-e^{-\kappa_r u}}{\kappa_r}, \tag{17}$$

$$\vartheta[u] \equiv \frac{1-e^{-\kappa_g u}}{\kappa_g}, \tag{18}$$

subject to the transversality condition that

$$\mu_r - \mu_g > \frac{\sigma_r^2}{2\,\kappa_r^2} - \frac{\sigma_r \sigma_y \rho_{r,y}}{\kappa_r} - \frac{\sigma_g \sigma_r \rho_{g,r}}{\kappa_g \kappa_r} - \Pi_y + \frac{\sigma_g^2}{2\kappa_g^2} + \frac{\sigma_g \sigma_y \rho_{g,y}}{\kappa_g}. \tag{19}$$

Thus, the model price for the market-portfolio or a stock is the summed value of a continuum of claims that each pay at a future time an amount respectively determined by the earnings process. The presence of an integral in (14) should not hamper the applicability of the model as the integral can be computed numerically.

The valuation formula in (14) is not as simple to comprehend as the Gordon dividend growth model. Realize that the Gordon model is a special case in which both G_t and r_t are constant over time: $G_t = g$ and $r_t = r$, for constants g and r. Consequently, both M_t and Y_t follow a geometric Brownian motion. In this case, we obtain $P_t = \frac{\alpha\,Y_t}{r+\Pi_y - g}$ provided $r + \Pi_y - g > 0$. In our economic setting, valuation is more complex as both discounting and cash flow forecasts have to be simultaneously assessed at the same time.

2.4 Dynamics of the Equity Premium

In deriving the valuation formula, we relied on a CAPM-like risk-return relation to arrive at the PDE in (8). In this sense, our model is consistent with and built upon developments in the risk-return literature. But, as seen, a risk-return equation alone is not sufficient to determine P_t since assumptions on the cash flow processes are also needed. Based on (2) and the pricing solution (14), we can show that the equity premium is,

$$\begin{aligned}
\mu_t - r_t &\equiv E_t\left(\frac{dP_t}{P_t}\right)/dt + \frac{\alpha\,Y_t}{P_t} - r_t, \\
&= -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dP_t}{P_t}\right)/dt, \\
&= \Pi_y \frac{Y_t}{P_t}\frac{\partial P_t}{\partial Y_t} + \Pi_g \frac{1}{P_t}\frac{\partial P_t}{\partial G_t} + \Pi_r \frac{1}{P_t}\frac{\partial P_t}{\partial r_t}, \qquad (20) \\
&= \Pi_y + \Pi_g \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \vartheta[u]\,du}{\int_0^\infty \overline{p}[t,u;G,r]\,du}\right) - \Pi_r \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \varrho[u]\,du}{\int_0^\infty \overline{p}[t,u;G,r]\,du}\right), \qquad (21)
\end{aligned}$$

where $\overline{p}[t,u;G,r]$ is displayed in (15). Equation (20) shows that the equity premium is a weighted sum of the risk premiums for shocks respectively due to earnings, expected earnings growth, and interest rate, with weights equal to the sensitivity of the price with respect to the respective state-variables.

Equation (21) follows from (20) since $\frac{Y_t}{P_t}\frac{\partial P_t}{\partial Y_t} = 1$, $\frac{\partial P_t}{\partial G_t} = \alpha\, Y_t \int_0^\infty \overline{p}[t,u;G,r] \times \vartheta[u]\, du$, and $\frac{\partial P_t}{\partial r_t} = -\alpha\, Y_t \int_0^\infty \overline{p}[t,u;G,r] \times \varrho[u]\, du$. Thus, the equilibrium equity premium is a function of the time-t interest rate, the expected EPS growth, the firm's required risk premiums, and the structural parameters governing the cash flow and interest rate processes. According to (21), $\mu_t - r_t$ is independent of the current level of cash flows and is mean-reverting with the state of r_t and G_t.

The dynamics of the state-variables under the equivalent martingale measure, Q, can facilitate our understanding of the nature of risk compensation in this economy. Based on (8), we may write the stock price as,

$$P_t \;=\; \alpha \int_t^\infty E_t^Q \left(e^{-\int_t^u r_s\, ds}\, Y_u \right) du, \tag{22}$$

where the processes for (Y_t, G_t, r_t) under the Q-measure are:

$$\frac{dY_t}{Y_t} \;=\; (G_t - \Pi_y)\, dt + \sigma_y\, d\widetilde{W}_t^y, \tag{23}$$

$$d\,G_t \;=\; \kappa_g \left([\mu_g^* - \Pi_g/\kappa_g] - G_t \right) dt + \sigma_g\, d\widetilde{W}_t^g, \tag{24}$$

$$d\,r_t \;=\; \kappa_r \left([\mu_r^* - \Pi_r/\kappa_r] - r_t \right) dt + \sigma_r\, d\widetilde{W}_t^r. \tag{25}$$

Economically, risk-averse investors seek to discount future cash flows more heavily under the equivalent martingale measure. For instance, we should expect $\Pi_r < 0$, which makes the drift of the risk-neutral discounting process higher. Consistent with this effect, a higher long-run mean $\mu_r = \mu_r - \Pi_r/\kappa_r$ will simultaneously reduce the discount bond price and raise all Treasury yields. Thus, our decomposition in (20) shows that $\Pi_r < 0$ can be expected to increase the overall equity premium, because $\frac{\partial P_t}{\partial r_t} < 0$. There is evidence from bond markets that the interest rate risk premium is non-zero (see, for example, Duffee (2002)).

A similar risk-aversion-based reasoning suggests that investors tend to be less optimistic about future cash flows under the equivalent martingale measure than under the physical probability measure. Intuitively, we have $\Pi_y > 0$ and $\Pi_g > 0$: the presence of both risk premiums decreases the drift of the (Y_t, G_t) process. The working of both of these forces reduces the present value of future cash flows and, thus, elevates the market risk premium. Thus, the earnings risk premium Π_y, the expected earnings growth risk premium Π_g, and the discounting risk premium receive positive compensation and contribute separately to the total equity premium.

To explore the properties of equity premium derived in (21), we turn to a comparative statics exercise and study how it responds to any structural parameter. In this example, $\kappa_r = 0.23$, $\mu_r^* = 7.8\%$, $\sigma_r = 0.012$, $\kappa_g = 1.44$, $\mu_g^* = 0.10$, $\sigma_g = 0.089$, σ_y=0.20, $\rho_{g,r}$= -0.05, $\rho_{g,y} = 1$, and $\alpha = 0.50$. We fix the interest rate risk premium $\Pi_r = -0.002$, the expected earnings growth risk premium $\Pi_g = 0.002$, and the earnings risk premium Π_y=0.06. In all calculations $r_t = 5.68\%$ and $G_t = 7.48\%$ which are market observed values as of July 1998 and correspond to S&P 500 index level of 1174.

Our numerical exercise shows that the equity premium is increasing in both G_t and μ_g^*, but decreasing in both r_t and μ_r^*. Therefore, as expected, positive shocks to expected EPS growth tend to raise the equity premium, whereas positive shocks to interest rates depress it. However, the equity premium is much more sensitive to μ_g^* (μ_r^*) than to G_t (r_t). Intuitively, these comparative static results hold because current expected EPS growth G_t may have a transitory component, whereas a change in μ_g is permanent. Lastly, the model equity premium increases with EPS growth volatility σ_y, the volatility of expected EPS growth σ_g, and the volatility of the interest rate σ_r. Risks as measured by these parameters raise the required compensation to shareholders. Modeling the EPS and the expected EPS processes explicitly indeed allows us to see how they affect the equity premium.

3 Time-Series Data on S&P 500 EPS, EPS Growth, and the Interest Rate

For the remainder of the paper we choose the S&P 500 index as the proxy for the market-portfolio. To explore whether the model equity premium derived in (21) is close to the sample equity premium requires three data inputs: expected EPS growth G_t, interest rate r_t, current EPS Y_t, and the model parameters. For the S&P 500 index, I/B/E/S did not start collecting analyst EPS estimates until January 1982. Thus, our focus is on the sample period from January 1982 to July 1998. Pastor and Stambaugh (2001) detect structural shifts in the equity premium especially over the past two decades. According to Lettau, Ludvigson, and Watcher (2004), the market price-to-earnings ratio rose sharply over this period and have argued in favor of the declining ex-ante equity risk premium explanation.

I/B/E/S US History File contains mid-month observations on reported actual earnings-per-share and consensus analyst forecasts of future S&P 500 earnings, plus the contemporaneous price. In implementation, I/B/E/S consensus analyst estimate for current-year S&P 500 EPS (i.e., FY1) is taken to be the proxy for Y_t. In any given month, the FY1 estimate may contain actual quarterly EPS numbers for the passed quarters of the fiscal year, with the EPS numbers for the remaining quarters being consensus analyst forecasts. Because firms' earnings typically exhibit seasonalities, the total EPS over a fiscal year is a natural proxy for Y_t.

Analyst-expected EPS growth from the current (FY1) to the next fiscal-year (FY2) is the measure for G_t. This choice is reasonable since the year-over-year EPS growth has been the conventional calculation method in the industry. For instance, quarter-over-quarter and month-over-month (if available) EPS growth rates would not be better proxies for G_t, as they would be subject to seasonal biases in earnings and revenue.

Valuation formulas for the market index and the equity premium also depend on interest rate r_t, for which there is no established benchmark. Empirically, movements in the 30-year Treasury yield are much more closely followed by stock market participants than the short-term rate, as the long-term yields often co-move strongly with S&P 500 earnings-yields. To be consistent with theory, however, we use the 3-month Treasury yield or those implied by the Kalman-filter as candidates for r_t in estimation and calibration. The 30-year Treasury yield is used in a robustness exercise. The source of monthly 3-month interest-rate is DataStream International, Inc.

To infer the interest rate risk premium independent of the price observations on the market portfolio, we rely on a panel of Treasury yields. We choose Treasury securities with constant maturity of 6 months, 2 years, 5 years, and 10 years. The Treasury yields are gathered from the Federal Reserve Board.

Table 1 reveals that the average equity premium over the sample period is 8.76% and volatile. Although the average equity premium is somewhat higher than the 7% reported by Mehra and Prescott (1985, 2003), it is nonetheless of a similar order of magnitude. That the equity index provides a higher return relative to bonds is also a stylized feature over our shorter sample.

Forward price-to-earnings ratio (the current price divided by FY1 earnings) has a sample average of 15.10, with a minimum price-to-earnings ratio of 7.28 and a maximum is 26.47. As seen, the average expected EPS growth for the S&P 500 index is 10.13% and varies between 0.09% and 26.13%. The average 3-month nominal interest rate is 6.28% with a standard deviation of 2.44%.

4 Implications of the Model for Equity Premium

The purpose of this section is two-fold. First, we pursue a traditional risk-based explanation of the equity premium puzzle and present an estimation strategy aimed at recovering each of the three components of the equity premium in (21). That is, we estimate Π_r, Π_g, Π_y, along with other model parameters, and judge empirical performance accordingly. Second, we quantitatively assess whether the risk premium parameterizations, interest rate dynamics, and cash flow dynamics embedded in the valuation model are capable of generating a reasonably large equity premium. We conduct these tasks while simultaneously fitting the Treasury yield curve as close as possible. Hence, our approach circumvents the risk-free rate puzzle outlined in Weil (1989).

4.1 How Large is the Interest Rate Risk Premium?

We first address the sign and magnitude of the interest rate risk premium by using the Kalman filtering approach and a panel of Treasury bond yields. This approach (i) enables the estimation of the interest rate risk premium jointly with the parameters of the interest rate dynamics in (7) (i.e., κ_r, μ_r^*, and σ_r), and (ii) allows us to test whether the interest rate model is able to generate realistic yield curve movements.

To implement this estimation procedure, we note that the transition equation for the instantaneous interest rate, r_t, can be expressed as (e.g., Bergstrom (1984)):

$$r_t = \mu_r^* (1 - e^{-\kappa_r \Delta t}) + e^{-\kappa_r \Delta t} r_{t-1} + \eta_t, \tag{26}$$

where $E_{t-1}[\eta_t] = 0$ and $E_{t-1}[\eta_t^2] = \sigma_r^2 \Delta t$, and η_t is a serially uncorrelated disturbance term that is distributed normal.

Next, let $\Psi_t = (\Psi_{j,t}, ..., \Psi_{J,t})'$ be the month-t observed Treasury yields where J denotes the number of yields employed in the estimation. As is standard from Babbs and Nowman (1999) and Chen and Scott (2003), the measurement equation describing observed Treasury yields is:

$$\Psi_t = \mathcal{U}_t + \mathcal{V}_t r_t + \upsilon_t, \qquad t = 1, ..., T, \tag{27}$$

where $\mathcal{U}_t$ is an $N \times 1$ vector with i-th element $\frac{\xi[\tau_i]}{\tau_i}$, $\mathcal{V}_t$ is an $N \times 1$ vector with i-th element $\frac{\varsigma[\tau_i]}{\tau_i}$, and $\upsilon_t \sim \mathcal{N}(0, \mathcal{H}_t)$. The normality of υ_t and η_t allows us to implement a Kalman filter recursion based on the maximum-likelihood approach described in Harvey (1991).

For this maximum-likelihood estimation, we select Treasury yields with maturity of 6 months, 2 years, 5 years, and 10 years and display the estimation results in Table 2. Panel A of this table shows that the interest rate parameters are reasonable and the interest-rate risk premium is in line with economic theory.

Let us discuss these parameter estimates in turn. First, the long-run interest rate, μ_r^*, is estimated at 7.28% and of an order of magnitude similar to that reported in Babbs and Nowman (1999) and Chen and Scott (2003). Second, the estimated $\kappa_r = 0.2313$ implies a half-life of 2.99 years, and indicates slow mean-reversion of the interest rate process. Third, the reported volatility of interest rate changes, $\sigma_r = 1.28\%$, suggests a relatively stable interest rate process. Finally, the maximized log-likelihood value for the estimation is 1804.93, and the estimated parameters are several times larger than their standard errors, suggesting statistical significance.

The estimated interest-rate risk premium, Π_r is, as we previously postulated, negative with a point estimate of -0.00201 (i.e., -20 basis points) and a standard error of 0.0005. Although the estimate appears quantitatively small, it can drive a substantial wedge between the risk-neutral and the physical interest rate processes. To see this point more clearly, we compute $\mu_r = \mu_r^* - \Pi_r/\kappa_r = 8.154\%$, which has the effect of raising the risk-neutral interest-rate drift by 86.9 basis points (hereafter, bp). Intuitively the risk factor $\Pi_r < 0$ causes a heavier discounting of future cash flows and theoretically supports the presence of a positive equity premium as the partial derivative of P_t with respect to the interest rate is negative in (21). Bonds provide a hedge during periods of stock market declines, which justifies a negative interest-rate risk premium. We refer the reader to the related work of Buraschi and Jitsov (2005) on the inflation risk premium and Bakshi and Chen (1996b) on a general model of inflation and interest rates in a monetary economy.

Goodness-of-fit statistics assessed in Panel B of Table 2 reveal that the interest rate model provides reasonable fitting-errors as measured by actual minus model-implied yield.

Across the Treasury yield curve the median absolute errors for 6-month, 2-year, 5-year, and 10-year yields are 37bp, 25bp, 35bp and 50bp, respectively. In sum, the time-series on the cross section of bond yields provide the desired flexibility in estimating the interest-rate risk premiums and the interest-rate parameters. Although there is scope for improvement, the pricing kernel process can realistically mimic both the short and the long end of the yield curve through time.

4.2 Maximum-Likelihood Estimation of the (Physical) G_t Process

The unavailability of contingent claims written directly on the G_t process precludes a joint estimation of the expected EPS growth processes in (5) and (24). We propose a two-step procedure to estimate Π_g. First, we exploit the transition density function to estimate the structural parameters, $\Theta_g \equiv \{\kappa_g, \mu_g^*, \sigma_g\}$, of the G_t process in (5). Second, taking Θ_g as given, we estimate Π_g, along with other unknown parameters, based on the time-series of S&P 500 index (the criterion function is specified in Section 4.3), and consequently recover the risk-neutral G_t process in (24).

Let $\{G_t : 1, \ldots, T\}$ be the monthly time-series on expected earnings growth rate. The discrete equation corresponding to the G_t process in (5), is:

$$G_t = \mu_g^* + e^{-\kappa_g} \left(G_{t-1} - \mu_g^*\right) + \zeta_t \tag{28}$$

where ζ_t is Gaussian mean-zero and satisfies the condition $E(\zeta_t \zeta_u) = 0$ for $t \neq u$, and

$$E(\zeta_t^2) = \frac{\sigma_g^2}{2\,\kappa_g} \left(1 - e^{-\kappa_g}\right). \tag{29}$$

Guided by Nowman (1997), we construct the likelihood function as minus twice the logarithmic of the Gaussian likelihood function

$$\max_{\kappa_g, \mu_g^*, \sigma_g} \quad \sum_{t=1}^{T} \left(\log\left\{ \frac{\sigma_g^2}{2\,\kappa_g} \left(1 - e^{-\kappa_g}\right) \right\} + \frac{\left\{ G_t - \mu_g^* - e^{-\kappa_g} \left(G_{t-1} - \mu_g^*\right) \right\}^2}{\left\{ \frac{\sigma_g^2}{2\,\kappa_g} \left(1 - e^{-\kappa_g}\right) \right\}^2} \right). \tag{30}$$

Maximizing the log-likelihood function in (30) by the choice of Θ_g, we report the maximum-likelihood parameter estimates below (the standard errors are shown in parenthesis):

$$\kappa_g = 1.4401 \ (0.4411) \tag{31}$$

$$\mu_g^* = 0.1024 \ (0.0153) \tag{32}$$

$$\sigma_g = 0.0894 \ (0.0047) \tag{33}$$

with an average log-likelihood value of 2.29575.

Several observations are relevant to our analysis. First, the point-estimate of long-run expected earnings growth rate, μ_g^*, is 10.04% and close to the sample average documented in Table 1. Thus, analysts have been optimistic about S&P 500 index earnings growth. Second, the volatility of changes in the expected earnings-per-share growth, σ_g, is 8.94%, which is considerably more volatile than the interest rate counterpart. Finally, according to the κ_g estimates, the S&P 500 expected earnings growth rate is mean-reverting with a half-life, $\log(2)/\kappa_g$, of 6 months. The duration of the expected earnings growth rate cycle is, thus, much shorter than the interest rate cycle and roughly consistent with stylized business cycle findings. Realizations of the physical G_t process are devoid of any information about the pricing measure, so the risk premium for expected earnings growth rate cannot be recovered through this estimation step.

4.3 Compensation for Cash Flow Risk and the Equity Premium

To estimate the risk premium for expected EPS growth risk, Π_g, and the risk premium for actual EPS growth, Π_y, and assess their implications for the equity premium, we make several choices. First, to reduce the estimation burden, we preset $\rho_{g,y} = 1$, and $\rho \equiv \rho_{g,r} = \rho_{r,y}$. This assumption implies that the actual and expected EPS growth rates are subject to a common random shock in (4) and (5). Second, we set Θ_g and $\{\kappa_r, \mu_r^*, \sigma_r, \Pi_r\}$ to the values estimated in Section 4.2 and Table 2, respectively. Thus, we treat these parameter inputs as representing the true values. Substituting Θ_g and $\{\kappa_r, \mu_r^*, \sigma_r, \Pi_r\}$ into (14)-(19), we can see that 5 parameters:

$$\Theta \equiv \{\Pi_g, \Pi_y, \alpha, \sigma_y, \rho\}, \tag{34}$$

are still required to determine the price of the market portfolio, P_t, in (14).

Observe that the valuation model for the market portfolio does not constitute a set of moment restrictions on asset prices; rather, it is an exact restriction on the price of the market portfolio in relation to the contemporaneous EPS, the expected EPS growth, and the interest rate. For this reason, the generalized method of moments and related econometric techniques may not be applicable.

Following the lead in fixed-income and option pricing, Θ is estimated using the time-series of market prices. We follow two estimation methods, one correcting, and the other not correcting, for the serial correlation of the model errors. Focusing on the first method, define from (14), the model price-to-earnings ratio as:

$$\mathrm{pe}_t \;\equiv\; \frac{P_t}{Y_t} = \alpha \int_0^\infty \overline{p}[t, u; G, r]\, du, \tag{35}$$

and let $\widetilde{\mathrm{pe}}_t$ be the month-t observed price-to-earnings ratio. Our estimation procedure tries to find a Θ to solve,

$$\mathrm{RMSE} \equiv \min_{\Theta} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \left(\alpha \int_0^\infty \overline{p}[t, u; G, r]\, du - \widetilde{\mathrm{pe}}_t \right)^2}, \tag{36}$$

subject to the transversality condition in (19). This estimation method seeks to minimize the sum of squared errors between each observed price-to-earnings ratio and the model-determined price-to-earnings ratio. The restriction in (19) ensures that P_t does not explode in each iteration of the minimization routine.

Fitting the price-to-earnings is desirable because P_t/Y_t serves as a normalized price that is comparable across time periods. If the purpose would be to fit the observed price levels as closely as possible, the estimation procedure would then favor the higher price observations. The criterion function in (36) fails to account for the serial correlation of the model pricing errors. However, when we assume a first-order autoregressive process for the model error, the resulting estimates are similar. Hence, we omit them and focus on the least-squares method in (36).

The optimized objective function value from (36), RMSE, is zero only if the obtained Θ estimate leads to a perfect fit of each market price-to-earnings by the model. In general, the average in-sample price-to-earnings pricing error will not be zero because the objective in (36) is to minimize the sum of squared errors, but not the average pricing errors.

In our estimation approach, the estimated risk premiums and parameters reflect the historical valuation standards applied to the S&P 500 index by the investors. Panel A of Table 3 reports the parameter estimates of Θ when the 3-month Treasury rate is used as the proxy for r_t. Consistent with how the market has priced the market-portfolio in the past, the market-implied ρ is negative with a ρ of -0.109. This mildly negative point estimate of ρ suggests that expected earnings growth rate is likely high when the interest rate is low, and vice-versa.

Another result worth emphasizing is that the dividend-payout ratio, α, is consistent with intuition: the estimated $\alpha = 0.41$ does not depart substantially from the historical average payout ratios of 44.29%. Table 3 also provides the estimate of $\sigma_y = 18.17\%$, with the conclusion that the cash flow process experiences high volatility.

One central observation from Table 3 is that the market-implied expected-EPS-growth risk premium, $\Pi_g = -0.145\%$, is surprisingly small relative to the market-implied earnings risk premium, $\Pi_y = 6.531\%$. For example, the reported Π_g, implies that the sample average of $\Pi_g \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \vartheta[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du} \right)$ is only 1 bp. This finding indicates that accounting for the compensation for bearing expected-EPS-growth risk plays virtually no role in explaining the equity premium puzzle.

If we accept the premise that the market fairly prices the S&P 500 index and correctly reflects the market price of various risks, then our empirical findings have a straightforward interpretation: Risk-averse agents may deem it unnecessary to "double-penalize" the physical drift of (Y_t, G_t) process. This may occur since P_t is homogenous of degree 1 in Y_t and has a first-order impact on the stock price. Therefore, a large compensation in the form of Π_y may make it unnecessary to require compensation for G_t risk. To further explain our reasoning, define $\widetilde{G}_t \equiv G_t - \lambda_y$. Therefore, we may write (23) and (24) as: $\frac{dY_t}{Y_t} = \widetilde{G}_t\, dt + \sigma_y\, d\widetilde{W}_t^y$, where $d\,\widetilde{G}_t = \left(\kappa_g \mu_g^* - \Pi_g - \kappa_g \Pi_y - \kappa_g \widetilde{G}_t\right) dt + \sigma_g\, d\widetilde{W}_t^g$. Thus, the presence of Π_y reduces the level and drift of the $\widetilde{G}_t$ process.

With $\Pi_r = -0.002$, the sample average of $-\Pi_r \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \varrho[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du} \right)$ is 77.16 bp. This suggests that accounting for discounting risk can help alleviate the equity premium puzzle.

Based on (21), the overall equity premium can, thus, be calculated as

$$\begin{aligned}
\mu_t - r_t &= \Pi_y + \Pi_g \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \vartheta[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du} \right) - \Pi_r \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \varrho[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du} \right), \\
&= 6.53\% + 0.01\% + 0.7716\%,
\end{aligned}$$

$$= \quad 7.31\%.$$

The ability of the model to generate an equity premium of 7.31% is in sharp contrast with the exercise in Mehra and Prescott (1985) that a standard representative agent model calibrated to the per-capita consumption data can generate at most a 0.40% equity premium. Thus, the proper parameterization of both the discounting structure and the cash flow process is key to improving performance by an asset pricing model and to achieving a reasonable equity premium. Our exercise in Panel B of Table 3 demonstrates that the equity premium is virtually insensitive to the choice of the interest rate in the estimation procedure in (36).

Another economic yardstick that can be applied is whether the estimated risk premiums and model parameters provide a "good enough" approximation of the market's implicit valuation process. In Table 3, we also present two percentage pricing-error measures, computed by dividing the market-to-model price difference by the market price: (i) the absolute percentage pricing error, and (ii) the mean percentage pricing error. The mean pricing error reflects the average pricing performance, while the absolute pricing error reflects the magnitude of the pricing errors as negative and positive errors do not cancel each other. According to the pricing-error measures, the model's fit is reasonable: the average mean pricing error is -7.22% with a standard deviation of 23.98%, and the absolute pricing error of the S&P 500's 18.30%. Given the negative sign of the average errors, the model price is on average higher than the market price.

In summary, the class of models examined here are not only consistent with the average equity premium and the term structure of interest rates, but also mimics the time-evolution of the S&P 500 index. The latter dimension imposes a stringent restriction on the validity of the pricing framework and differentiates this paper from other studies on the equity premium.

5 Concluding Remarks and Extensions

The equity premium puzzle advocated by Mehra and Prescott (1985) remains a fascinating problem awaiting new and novel answers. This paper investigated the impact of cash flow risk and discounting risk on the aggregate equity premium, the price of the market portfolio, and the default-free bond prices. Our theoretical approach is based on the observation that aggregate per-capita consumption is hard to measure empirically. Thus, if we can replace the empirically difficult-to-estimate marginal utility by a pricing-kernel function of observables and then specify both the primitive process for discounting and the exogenous cash flow stream, we will have an equilibrium asset pricing model based on observable state variables. Once this is done we can endogenously solve for the equity premium, the price of the market portfolio and the term structure of interest rates within the same underlying equilibrium.

Embedded in the closed-form solution for the market portfolio and the bond prices are compensations for cash flow risk and discounting risk. With the solution for the risk premium explicitly given, we can then estimate the model to evaluate its empirical performance. This approach allows us to avoid the impact of unobservable consumption on inferences regarding the model's performance. Our illustrative model is based on the assumption that aggregate dividend equals a fixed fraction of aggregate earnings plus noise, and the expected aggregate earnings growth follows a mean-reverting stochastic process. Moreover, the economy-wide pricing kernel is chosen to be consistent with (i) a constant market price of aggregate risk and (ii) a mean-reverting interest rate process with constant volatility.

S&P 500 index-based estimation results show that the framework is quantitatively useful in explaining the observed market equity premium. Specifically, we find that the interest rate risk premium is negative and the cash flow risk premium is positive. Overall, disentangling the equity premium into its cash flow and discounting components produces an economically meaningful equity premium of 7.31%.

Our empirical results suggest three possible avenues for theoretical research. First, one can introduce richer cash flow dynamics and interest rate dynamics that possess stochastic volatility. Having multi-dimensional structures for the state variables with priced volatility risks can lead to more realistic models for the market portfolio and the equity premium. Second, one can examine alternative risk premium specifications that allow for richer stochastic variation in the risk premiums. Third, the valuation model can be used to pin down the sources of market return predictability, as in Menzly, Santos, and Veronesi (2004).

The equity premium puzzle occupies a special place in the theory of finance and economics, and more progress is needed to understand the spread of equities over bonds. Determining the factors that drive the equity premium over time, and across countries, will likely remain an active research agenda.

Appendix

To derive the analytical solution to the market portfolio, we note from equations (1) and (3) that P_t solves,

$$P_t = \alpha \int_t^\infty E_t\left[\frac{M_u}{M_t} Y_u\right] du, \tag{37}$$

since dZ_t is uncorrelated with dM_t. We also require by the transversality condition that $P_t < \infty$ for all t, which is the condition that the price of the market portfolio remain bounded for all pricing kernel and cash flow processes.

Inserting the pricing kernel process (6) into (37) and using the earnings process (4)-(5), we note, by the Markov property, that P_t can only be a function of Y_t, r_t, and G_t. Write $P[Y_t, G_t, r_t]$, where the interest rate process is as specified in (7). Therefore, the dynamics of the market portfolio, by Ito's lemma, is given by:

$$\begin{aligned} dP_t &= \frac{1}{2}\frac{\partial^2 P}{\partial Y^2}(dY)^2 + \frac{\partial P}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 P}{\partial G^2}(dG)^2 + \frac{\partial P}{\partial G} dG + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr)^2 + \frac{\partial P}{\partial r} dr \\ &\quad + \frac{\partial^2 P}{\partial Y \partial G} dY\, dG + \frac{\partial^2 P}{\partial Y \partial r} dY\, dr + \frac{\partial^2 P}{\partial G \partial r} dr\, dG. \end{aligned} \tag{38}$$

Substituting (38) into (2) implies that the instantaneous equity premium is,

$$\begin{aligned} \mu_t - r_t &= -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dP_t}{P_t}\right)/dt, \\ &= -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{1}{P_t}\frac{\partial P}{\partial Y} dY + \frac{1}{P_t}\frac{\partial P}{\partial G} dG + \frac{1}{P_t}\frac{\partial P}{\partial r} dr\right)/dt, \end{aligned} \tag{39}$$

where the instantaneous expected return is, $\mu_t = E_t\left[\frac{dP_t}{P_t}\right]/dt + \frac{\alpha Y_t}{P_t}$.

Relying on (38) and taking expectations, we may obtain,

$$\begin{aligned} E_t\left[\frac{dP_t}{P_t}\right] &= \frac{1}{2}\frac{1}{P_t}\frac{\partial^2 P}{\partial Y^2} E_t[dY^2] + \frac{1}{P_t}\frac{\partial P}{\partial Y} E_t[dY] + \frac{1}{2}\frac{1}{P_t}\frac{\partial^2 P}{\partial G^2} E_t[dG]^2 + \frac{1}{P_t}\frac{\partial P}{\partial G} E_t[dG] \\ &\quad + \frac{1}{2}\frac{1}{P_t}\frac{\partial^2 P}{\partial r^2} E_t[dr^2] + \frac{1}{P_t}\frac{\partial P}{\partial r} E_t[dr] \\ &\quad + \frac{1}{P_t}\frac{\partial^2 P}{\partial Y \partial G} E_t[dY\, dG] + \frac{1}{P_t}\frac{\partial^2 P}{\partial Y \partial r} E_t[dY\, dr] + \frac{1}{P_t}\frac{\partial^2 P}{\partial G \partial r} E_t[dr\, dG]. \end{aligned} \tag{40}$$

Combining the expressions in (39) and (40) and using the definition of the instantaneous expected rate of return, we have

$$
\begin{aligned}
& \frac{1}{2}\frac{\partial^2 P}{\partial Y^2}E_t[dY^2] + \frac{\partial P}{\partial Y}E_t[dY] + \frac{1}{2}\frac{\partial^2 P}{\partial G^2}E_t[dG]^2 + \frac{\partial P}{\partial G}E_t[dG] + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}E_t[dr^2] \\
& + \frac{\partial P}{\partial r}E_t[dr] + \frac{\partial^2 P}{\partial Y \partial G}E_t[dY\,dG] + \frac{\partial^2 P}{\partial Y \partial r}E_t[dY\,dr] + \frac{\partial^2 P}{\partial G \partial r}E_t[dr dG] \\
& - r\,P\,dt + \alpha\,Y\,dt \\
= \; & -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{\partial P}{\partial Y}dY + \frac{\partial P}{\partial G}dG + \frac{\partial P}{\partial r}dr\right). \qquad (41)
\end{aligned}
$$

Based on (41), now define the risk premium for the earnings shocks, expected earnings growth, and interest rate, respectively, as:

$$
\begin{aligned}
\Pi_y & \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dY_t}{Y_t}\right)/dt, \\
\Pi_g & \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, dG_t\right)/dt, \\
\Pi_r & \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, dr_t\right)/dt.
\end{aligned}
$$

This immediately implies that,

$$
\begin{aligned}
& \frac{1}{2}\frac{\partial^2 P}{\partial Y^2}E_t[dY^2] + \frac{\partial P}{\partial Y}E_t[dY] + \frac{1}{2}\frac{\partial^2 P}{\partial G^2}E_t[dG]^2 + \frac{\partial P}{\partial G}E_t[dG] + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}E_t[dr^2] \\
& + \frac{\partial P}{\partial r}E_t[dr] + \frac{\partial^2 P}{\partial Y \partial G}E_t[dY\,dG] + \frac{\partial^2 P}{\partial Y \partial r}E_t[dY\,dr] + \frac{\partial^2 P}{\partial G \partial r}E_t[dr dG] - r\,P\,dt + \alpha\,Y\,dt \\
= \; & \frac{\partial P}{\partial Y}Y\,\Pi_y\,dt + \frac{\partial P}{\partial G}\Pi_g\,dt + \frac{\partial P}{\partial r}\Pi_r\,dt. \qquad (42)
\end{aligned}
$$

Simplifying this equation and using the dynamics for Y_t, G_t and r_t, leads to the following partial differential equation for P_t:

$$
\begin{aligned}
& \frac{1}{2}\sigma_y^2 Y^2 \frac{\partial^2 P}{\partial Y^2} + (G - \Pi_y)\,Y\frac{\partial P}{\partial Y} + \rho_{g,y}\sigma_y\sigma_g\,Y\frac{\partial^2 P}{\partial Y \partial G} + \rho_{r,y}\sigma_y\sigma_r\,Y\frac{\partial^2 P}{\partial Y \partial r} + \\
& \rho_{g,r}\sigma_g\sigma_r\,\frac{\partial^2 P}{\partial G \partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2} + \kappa_r\,(\mu_r - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_g^2\frac{\partial^2 P}{\partial G^2} \\
& + \kappa_g\,(\mu_g - G)\frac{\partial P}{\partial G} - r\,P + \alpha\,Y = 0, \qquad (43)
\end{aligned}
$$

and must be solved subject the restriction that $P_t < \infty$. In the valuation partial differential equation (43) we have set, $\mu_g = \mu_g^* - \frac{\Pi_g}{\kappa_g}$ and $\mu_r \equiv \mu_r^* - \frac{\Pi_r}{\kappa_r}$. Consider the following candidate solution,

$$P_t = \alpha \int_0^\infty \hat{p}[t, u; Y, G, r]\, du. \tag{44}$$

Clearly, $\hat{p}[t+u, 0; Y, G, r] = Y_{t+u}$. Thus, we have the partial differential equation for $\hat{p}[t, u; Y, G, r]$ as,

$$\begin{aligned}
&(G - \Pi_y)\, Y\, \frac{\partial \hat{p}}{\partial Y} + \rho_{g,y}\sigma_y\sigma_g\, Y\, \frac{\partial^2 \hat{p}}{\partial Y\, \partial G} + \rho_{r,y}\sigma_y\sigma_r\, Y\, \frac{\partial^2 \hat{p}}{\partial Y\, \partial r} + \\
&\rho_{g,r}\sigma_g\sigma_r\, \frac{\partial^2 \hat{p}}{\partial G \partial r} + \frac{1}{2}\sigma_r^2\, \frac{\partial^2 \hat{p}}{\partial r^2} + \kappa_r\,(\mu_r - r)\, \frac{\partial \hat{p}}{\partial r} + \frac{1}{2}\sigma_g^2 \frac{\partial^2 \hat{p}}{\partial G^2} \\
&+ \kappa_g\,(\mu_g - G)\, \frac{\partial \hat{p}}{\partial G} - r\, \bar{p} - \frac{\partial \hat{p}}{\partial u} = 0.
\end{aligned} \tag{45}$$

Suppose $\hat{p}[t, u; G, r] = Y_t\, \exp\left(\varphi[u] - \varrho[u]\, r_t + \vartheta[u]\, G_t\right)$. Taking the required partial derivatives with respect to Y_t, G_t and r_t and solving the valuation equations lead to a set of ordinary differential equations. Solving the ordinary differential equations subject to the boundary conditions $\varphi[0] = 0$, $\varrho[0] = 0$ and $\vartheta[0] = 0$ yields (14)-(15). The transversality condition (19) ensures that the restriction $\varphi[0] = 0$ is satisfied. □

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Table 1: Equity Premium for S&P 500 Index (January 1982 to July 1998)

The sample period is January 1982 to July 1998 with 199 monthly observations. The expected earnings-per-share growth for S&P 500 index, G_t, is the consensus earnings-per-share forecast for FY2 divided by FY1, minus 1. The price-to-earnings ratio, P/E, is the current S&P 500 index level normalized by FY1 earnings-per-share. We report the average, the standard deviation, the maximum, and the minimum. The computation of the monthly equity premium is based on the 3-month interest rate. The earnings and price on S&P 500 is collected from I/B/E/S and the interest rates are from the Federal Reserve Board.

	Average	**Std.**	**Max.**	**Min.**
Price-to-Earnings Ratio	15.10	4.13	26.47	7.28
Expected Earnings Growth	10.13%	5.31%	26.13%	0.09%
Interest Rate	6.98%	2.13%	14.68%	5.68%
Monthly Equity Premium	0.0073	0.040	0.162	-0.200

Table 2: Interest Rate Risk Premium Based on Kalman Filtering Estimation

The reported parameters of the interest rate process and the interest rate risk premium are based on Kalman filtering. We specify the interest rate process under the physical probability measure as:

$$dr_t \quad = \quad (\kappa_r \, \mu_r - \kappa_r \, r_t \,) \, dt + \sigma_r \, dW_t^r,$$

and under the equivalent martingale measure as

$$dr_t \quad = \quad (\kappa_r \, \mu_r - \Pi_r - \kappa_r \, r_t \,) \, dt + \sigma_r \, d\widetilde{W}_t^r,$$

The estimation uses a monthly time-series of treasury yields with maturity of 6-months, 2-years, 5-years and 10-years. The asymptotic standard errors are in parenthesis, and based on the outer-product of the log-likelihood function. Maximized log-likelihood function is reported as Log-Lik. Panel B reports the median absolute pricing errors (in bp), and the root mean squared pricing errors (in bp).

Panel A: Parameter Estimates

Parameter	κ_r	σ_r	μ_r^*	Π_r	Log-Lik
r_t process	0.2313 (0.0135)	0.0128 (0.0008)	0.0728 (0.0022)	-0.0020 (0.0005)	1804.93

Panel B: Fitting Errors for Bonds

	6-months	**2-years**	**5-years**	**10-years**
Median Absolute Pricing Errors (bp)	37	25	35	50
Squared-root of Mean Squared Errors (bp)	48	33	44	59

Table 3: Estimation of Risk Premiums for Earnings Growth and Expected Earnings Growth Rate: Implications for Equity Premium

Estimation of the risk premiums is based on S&P 500 index observations from January 1982 to July 1998 (199 observations). We minimize the distance between the model price-to-earnings ratio and the market price-to-earnings ratio denoted by $\widetilde{\mathrm{pe}}_t$:

$$\mathrm{RMSE} \equiv \min_{\Theta} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \left(\alpha \int_0^{\infty} \overline{p}[t, u; G, r] \, du - \widetilde{\mathrm{pe}}_t \right)^2},$$

subject to the transversality condition $\mu_r - \mu_g > \frac{\sigma_r^2}{2\,\kappa_r^2} + \frac{\sigma_g^2}{2\kappa_g^2} + \frac{\sigma_g \sigma_y \rho_{g,y}}{\kappa_g} - \frac{\sigma_r \sigma_y \rho_{r,y}}{\kappa_r} - \frac{\sigma_g \sigma_r \rho_{g,r}}{\kappa_g \kappa_r} - \Pi_y$. In this estimation $\kappa_r = 0.2313$, $\sigma_r = 0.0128$, $\mu_r^* = 0.0728$ and $\lambda_r = -0.00201$ which are based on the results in Table 2, and $\rho_{g,y} = 1$, and $\rho \equiv \rho_{g,r} = \rho_{r,y}$. Parameters governing the dynamics of the expected earnings growth rate are fixed to $\kappa_g = 1.4401$, $\mu_g^* = 0.1024$, and $\sigma_g = 0.089$. We compute the model error $\epsilon_t \equiv Y_t \left(\alpha \int_0^{\infty} \overline{p}[t, u; G, r] \, du - \widetilde{\mathrm{pe}}_t \right)$, and report the average pricing errors and the average absolute pricing errors. The standard deviations are shown as Std(.). Each month we compute the model equity premium as $\mu_t - r_t = \Pi_y + \Pi_g \left(\frac{\int_0^{\infty} \overline{p}[t,u;G,r] \times \vartheta[u] \, du}{\int_0^{\infty} \overline{p}[t,u;G,r] \, du} \right) - \Pi_r \left(\frac{\int_0^{\infty} \overline{p}[t,u;G,r] \times \varrho[u] \, du}{\int_0^{\infty} \overline{p}[t,u;G,r] \, du} \right)$, and report the sample average as Mean($\mu_t - r_t$), All calculations in Panel A are done using the 3-month treasury rate as a proxy for the interest rate, and repeated in Panel B using the 30-year treasury rate.

Panel A: Estimation Based on 3-Month Treasury Rate

Π_g	Π_y	α	σ_y	ρ	RMSE	Mean(ϵ_t) {Std(ϵ_t)}	Mean($\|\epsilon_t\|$) {Std($\|\epsilon_t\|$)}	Mean($\mu_t - r_t$)
0.001450	0.06531	0.4100	0.1817	-0.109	3.2293	-7.22% {23.98%}	18.30% {17.63}	7.312%

Panel B: Estimation Based on 30-Year Treasury Yield

Π_g	Π_y	α	σ_y	ρ	RMSE	Mean(ϵ_t) {Std(ϵ_t)}	Mean($\|\epsilon_t\|$) {Std($\|\epsilon_t\|$)}	Mean($\mu_t - r_t$)
0.001145	0.06379	0.4744	0.1513	-0.074	3.1351	-7.62% {23.66%}	19.05% {15.92}	7.213%